Multiplicative operators in the spaces of Schwartz families

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Abstract

In this paper we introduce and study the multiplication among smooth functions and Schwartz families. This multiplication is fundamental in the formulation and development of a spectral theory for Schwartz linear operators in distribution spaces, to introduce efficiently the Schwartz eigenfamilies of such operators and to build up a functional calculus for them. The definition of eigenfamily is absolutely natural and this new operation allows us to develop a rigorous and manageable spectral theory for Quantum Mechanics, since it appears in a form extremely similar to the current use in Physics.

1 Introduction

In the Spectral Theory of $^{\mathcal{S}}$linear operators, the eigenvalues corresponding to the elements of certain $^{\mathcal{S}}$families have fundamental importance. If L is an $^{\mathcal{S}}$linear operator and v is an $^{\mathcal{S}}$family, the family v is defined an eigenfamily of the operator L if there exists a real or complex function l - defined on the set of indices of the family v - such that the relation

$$L(v_p) = l(p)v_p,$$

holds for every index p of the family v. As we already have seen, in the context of $^{\mathcal{S}}$linear operators, it is important how the operator L acts on the entire family v. Taking into account the above definition, it is natural to consider the image family $L(v)$ as the product - in pointwise sense - of the family v by the function l, but:

- *is the pointwise multiplication an operation in the space of $^{\mathcal{S}}$families?*
- *what kind of properties are satisfied by this product?*

In this chapter we define and study the properties of such product.

2 $\mathcal{O}_M$ Functions

We recall, for convenience of the reader, some basic notions from theory of distributions.

Definition (of slowly increasing smooth function). *We denote by $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$, or more simply by $\mathcal{O}_M^{(n)}$, the subspace of all smooth functions f, belonging to the space $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{K})$, such that, for every test function $\phi \in \mathcal{S}_n$ the product ϕf lives in $\mathcal{S}_n$. The space $\mathcal{O}_M(\mathbb{R}^n, \mathbb{K})$ is said to be the* ***space of smooth functions from $\mathbb{R}^n$ into the field $\mathbb{K}$ slowly increasing at infinity (with all their derivatives)****.*

In other terms, the functions f belonging to the space $\mathcal{O}_M^{(n)}$ are the only smooth functions which can generate a multiplication operator

$$M_f : \mathcal{S}_n \to \mathcal{S}_n$$

of the space $\mathcal{S}_n$ into the space $\mathcal{S}_n$ itself, (obviously) by the relation

$$M_f(g) = fg.$$

This is the motivation of the importance of these functions in Distribution Theory, and the symbol itself $\mathcal{O}_M$ depends on this fact ($\mathcal{O}_M$ stands for multiplicative operators).

Let us see a first characterization.

Proposition. *Let $f \in \mathcal{E}_n$ be a smooth function. Then the following conditions are equivalent:*

1) *for all multi-index $p \in \mathbb{N}_0^n$ there is a polynomial P_p such that, for any point $x \in \mathbb{R}^n$, the following inequality holds*

$$|\partial^p f(x)| \leq |P_p(x)|\,;$$

2) *for any test function $\phi \in \mathcal{S}_n$ the product ϕf lies in $\mathcal{S}_n$;*

3) *for every multi-index $p \in \mathbb{N}_0^n$ and for every test function $\phi \in \mathcal{S}_n$ the product $(\partial^p f)\,\phi$ is bounded in $\mathbb{R}^n$.*

2.1 Topology

The standard topology of the space $\mathcal{O}_M^{(n)}$ is the locally convex topology defined by the family of seminorms

$$\gamma_{\phi,p}(\phi) = \sup_{x\in\mathbb{R}^n} |\phi(x)\partial^p f(x)|$$

with $\phi \in \mathcal{S}_n$ and $p \in \mathbb{N}_0^n$. This topology does not have a countable basis. Also, it can be shown that the space $\mathcal{O}_M^{(n)}$ is a complete space. A sequence $(f_j)_{j\in\mathbb{N}}$ converges to zero in $\mathcal{O}_M^{(n)}$ if and only if for every test function $\phi \in \mathcal{S}_n$ and for every multi-index $p \in \mathbb{N}_0^n$, the sequence of functions $(\phi\partial^p f_j)_{j\in\mathbb{N}}$ converges to zero uniformly on $\mathbb{R}^n$; or, equivalently, if, for every test function $\phi \in \mathcal{S}_n$, the sequence $(\phi f_j)_{j\in\mathbb{N}}$ converges to zero in $\mathcal{S}_n$. A filter F on $\mathcal{O}_M^{(n)}$ converges to zero in $\mathcal{O}_M^{(n)}$ if and only if for every test function $\phi \in \mathcal{S}_n$, the filter ϕF converges to zero in $\mathcal{S}_n$.

2.2 Bounded sets in $\mathcal{O}_M^{(n)}$

A subset B of $\mathcal{O}_M^{(n)}$ is bounded (in the topological vector space $\mathcal{O}_M^{(n)}$) if and only if, for all multi-index $p \in \mathbb{N}_0^n$, there is a polynomial P_p such that, for any function $f \in B$, the following inequality holds true

$$|\partial^p f(x)| \leq P_p(x),$$

for any point $x \in \mathbb{R}^n$.

2.3 Multiplication in $\mathcal{S}_n$ by $\mathcal{O}_M^{(n)}$ functions

The bilinear map

$$\Phi : \mathcal{O}_M^{(n)} \times \mathcal{S}_n \to \mathcal{S}_n : (\phi, f) \mapsto \phi f$$

is separately continuous with respect to the usual topologies of the spaces $\mathcal{O}_M^{(n)}$ and $\mathcal{S}_n$. It follows immediately that the multiplication operator M_f, associated with an $^{\mathcal{O}_M}$function f, is continuous (with respect to the standard topology of the Schwartz space $\mathcal{S}_n$). Moreover, the transpose of the operator M_f is the operator

$$^tM_f : \mathcal{S}'_n \to \mathcal{S}'_n$$

defined by

$$\begin{aligned} {}^tM_f(u)(g) &= u(M_f(g)) = \\ &= u(fg) = \\ &= fu(g), \end{aligned}$$

for every u in $\mathcal{S}'_n$ and for every g in $\mathcal{S}_n$. So that, the transpose of the multiplication M_f is the multiplication on $\mathcal{S}'_n$ by the function f. Indeed, the multiplication of a tempered distribution by an ${}^{\mathcal{O}_M}$function is defined by the transpose of M_f, since this last operator is self-adjoint with respect to the canonical bilinear form on $\mathcal{S}_n \times \mathcal{S}_n$. In fact, obviously, we have

$$\langle M_f(g), h\rangle = \langle g, M_f(h)\rangle,$$

for every pair (g, h) in that Cartesian product $\mathcal{S}_n \times \mathcal{S}_n$. So we can use the standard procedure to extend regular operators (operators admitting an adjoint with respect to the standard bilinear form) from their domain $\mathcal{S}_n$ to the entire space $\mathcal{S}'_n$.

2.4 ${}^{\mathcal{S}}$Family of the multiplication operator M_f

Since the multiplication operator $M_f : \mathcal{S}_n \to \mathcal{S}_n$ is continuous, we can associate with it an ${}^{\mathcal{S}}$family v, in the canonical way. We have

$$\begin{aligned} v_p &= (M_f^{\vee})_p = \\ &= \delta_p \circ M_f = \\ &= {}^t M_f(\delta_p) = \\ &= f\delta_p = \\ &= f(p)\delta_p, \end{aligned}$$

for every p in $\mathbb{R}^n$. In the language of Schwartz matrices we can say that to the operator M_f is associated the Schwartz diagonal matrix $f\delta$.

3 Product in $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ by ${}^{\mathcal{O}_M}$functions

The basic remark is the following.

Proposition. *Let $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ be a continuous linear operator and let f be a function of class $\mathcal{O}_M^{(m)}$. Then, the mapping*

$$fA : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto fA(\phi)$$

is a linear and continuous operator too; it is indeed the composition

$$M_f \circ A,$$

where M_f is the multiplication operator on $\mathcal{S}_m$ by the function f.

Proof. It is absolutely straightforward. First of all we note that the product fA is well defined. In fact, we have

$$(fA)(\phi) = fA(\phi),$$

and the right-hand function lies in the space $\mathcal{S}_m$ because the function f lies in the space $\mathcal{O}_M^{(m)}$ and the function $A(\phi)$ lies in the space $\mathcal{S}_m$. Moreover, the bilinear application

$$\Phi : \mathcal{O}_M^{(m)} \times \mathcal{S}_m \to \mathcal{S}_m : (f, \psi) \mapsto f\psi$$

is separately continuous and we have

$$\begin{aligned} (fA)(\phi) &= fA(\phi) = \\ &= \Phi(f, A(\phi)) = \\ &= M_f(A(\phi)), \end{aligned}$$

i.e.,

$$\begin{aligned} fA &= \Phi(f, \cdot) \circ A = \\ &= M_f \circ A, \end{aligned}$$

hence the operator fA is the composition of two linear continuous maps and then it is a linear and continuous operator. ■

Definition. *Let $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ and $f \in \mathcal{O}_M^{(m)}$. The operator*

$$fA : \mathcal{S}_n \to \mathcal{S}_m : \phi \mapsto fA(\phi)$$

*is called **the product of the operator** A **by the function** f.*

Proposition. *Let $A, B \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$ be two continuous linear operators and f, g be two functions in $\mathcal{O}_M^{(m)}$. Then, we have*

1) $(f + g)A = fA + gA$; $f(A + B) = fA + fB$; $1_{\mathbb{R}^m} A = A$, where the function $1_{\mathbb{R}^m}$ is the constant function of $\mathbb{R}^m$ into $\mathbb{K}$ with value 1;

2) the map

$$\Phi : \mathcal{O}_M^{(m)} \times \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m) : (f, A) \mapsto fA$$

is a bilinear map.

Proof. It's a straightforward computation. ■

The above bilinear application is called *multiplication of operators by $\mathcal{O}_M$ functions.*

3.1 The algebra $\mathcal{O}_M^{(m)}$

It's easy to see that the algebraic structure $(\mathcal{O}_M^{(m)}, +, \cdot)$ is a commutative ring with identity, with respect to the usual pointwise addition and multiplications. For instance, the multiplication is the operation

$$\cdot\ : \mathcal{O}_M^{(m)} \times \mathcal{O}_M^{(m)} \to \mathcal{O}_M^{(m)} : (f, g) \mapsto fg,$$

where, obviously, if $f, g \in \mathcal{O}_M^{(m)}$, then the pointwise product fg still lies in $\mathcal{O}_M^{(m)}$. The identity of the ring is the function $1_m := 1_{\mathbb{R}^m}$. Moreover, we have that the subspace $\mathcal{S}_m$ of the space $\mathcal{O}_M^{(m)}$ is an ideal of the ring $\mathcal{O}_M^{(m)}$. The subring of $\mathcal{O}_M^{(m)}$ formed by the invertible elements of $\mathcal{O}_M^{(m)}$ is exactly the multiplicative subgroup of those elements f such that the multiplicative inverse f^{-1} belongs to the space $\mathcal{O}_M^{(m)}$ too.

So that, the space $\mathcal{O}_M^{(m)}$ is a locally convex topological algebra with unit element.

3.2 The module $\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$

Proposition. *Let $\cdot$ be the multiplication by $\mathcal{O}_M^{(m)}$ functions defined in the above theorem. Then, the algebraic structure $(\mathcal{L}(\mathcal{S}_n, \mathcal{S}_m), +, \cdot)$ is a left module over the ring $(\mathcal{O}_M^{(m)}, +, \cdot)$.*

Proof. Recalling the preceding theorem, we have to prove only the pseudo-associative law, i.e. we have to prove that for every couple of functions $f, g \in \mathcal{O}_M^{(m)}$ and for every linear continuous operator $A \in \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$, we have

$$(fg)A = f(gA).$$

In fact, for each $\phi \in \mathcal{S}_n$, we have

$$\begin{aligned} [(fg)A](\phi) &= (fg)A(\phi) = \\ &= f(gA(\phi)) = \\ &= f(gA)(\phi)) = \\ &= [f(gA)](\phi), \end{aligned}$$

as we desired. ■

4 Products of ${}^{\mathcal{S}}$families by ${}^{\mathcal{O}_M}$functions

The central definition of the chapter is the following.

Definition (product of Schwartz families by smooth functions). *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$family of distributions and let $f \in \mathcal{C}^\infty(\mathbb{R}^m, \mathbb{K})$ be a smooth function. The* ***product of the family*** *v* ***by the function*** *f is the family*

$$fv := (f(p)v_p)_{p \in \mathbb{R}^m}.$$

Theorem. *Let $v \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$family and $f \in \mathcal{O}_M^{(m)}$. Then, the family fv lies in $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Moreover, we have*

$$(fv)^\wedge = f\widehat{v}.$$

Consequently, concerning the superposition operator of the family fv, since $f\widehat{v} = M_f \circ \widehat{v}$, we have

$${}^t(fv)^\wedge = {}^t\widehat{v} \circ {}^tM_f,$$

or equivalently, in superposition form

$$\int_{\mathbb{R}^m} a(fv) = \int_{\mathbb{R}^m} (fa)v,$$

for every coefficient distribution a in $\mathcal{S}'_m$.

Proof. Let $\phi \in \mathcal{S}_n$ be a test function, we have

$$\begin{aligned} (fv)(\phi)(p) &= (fv)_p(\phi) = \\ &= (f(p)v_p)(\phi) = \\ &= f(p)v_p(\phi) = \\ &= f(p)\widehat{v}(\phi)(p) \end{aligned}$$

and hence the function $(fv)(\phi)$ equals $f\widehat{v}(\phi)$, which lies in $\mathcal{S}_m$. Thus, the product fv lies in the space of Schwartz families $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. For any test function $\phi \in \mathcal{S}_n$, by the above consideration, we deduce

$$(fv)^\wedge(\phi) = f\widehat{v}(\phi),$$

that is, the equality of operators

$$(fv)^\wedge = f\widehat{v},$$

where $f\widehat{v}$ is the product of the operator $\widehat{v}$ by the function f, product which belongs to the space $\mathcal{L}(\mathcal{S}_m, \mathcal{S}_n)$. Moreover, concerning the superposition operator of the family fv, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^m} a(fv) &= {}^t(fv)^\wedge(a) = \\
&= {}^t(f\widehat{v})(a) = \\
&= {}^t(M_f \circ \widehat{v})(a) = \\
&= ({}^t\widehat{v} \circ {}^tM_f)(a) = \\
&= {}^t\widehat{v}({}^tM_f(a)) = \\
&= {}^t\widehat{v}(fa) = \\
&= \int_{\mathbb{R}^m} (fa)v,
\end{aligned}$$

for every distribution a in $\mathcal{S}'_m$. ■

Theorem. *Let f, g two functions in the space $\mathcal{O}_M^{(m)}$ and v, w two Schwartz families in the space $\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$. Then, we have:*

1) $(f+g)v = fv + gv$, $f(v+w) = fv + fw$ *and* $1_m v = v$;

2) *the map*

$$\Phi : \mathcal{O}_M^{(m)} \times \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) : (f, v) \mapsto fv$$

is a bilinear map.

Proof. 1) For all $p \in \mathbb{R}^m$, we have

$$\begin{aligned}
[(f+g)\,v]\,(p) &= (f+g)(p)v_p = \\
&= (f(p) + g(p))v_p = \\
&= f(p)v_p + g(p)v_p = \\
&= (fv)_p + (gv)_p,
\end{aligned}$$

i.e. $(f+g)v = fv + gv$. For all $p \in \mathbb{R}^m$, we have

$$\begin{aligned}
[f\,(v+w)]\,(p) &= f(p)(v+w)_p = \\
&= f(p)(v_p + w_p) = \\
&= f(p)v_p + f(p)w_p = \\
&= (fv)_p + (fw)_p,
\end{aligned}$$

i.e. $f(v+w) = fv + fw$. For all $p \in \mathbb{R}^m$, we have

$$(1_{\mathbb{R}^m} v)(p) = 1_{\mathbb{R}^m}(p)v_p = v_p;$$

i.e. $1_{\mathbb{R}^m} v = v$. 2) follows immediately by 1). ■

The bilinear application of the point 2) of the preceding theorem is called *multiplication of Schwartz families by $\mathcal{O}_M$ functions.*

Theorem (of structure). *Let $\cdot$ the operation defined above. Then, the algebraic structure $(\mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n), +, \cdot)$ is a left module over the ring $(\mathcal{O}_M^{(m)}, +, \cdot)$.*

Proof. It's analogous to the proof of the corresponding proposition for operators. ■

Theorem (of isomorphism). *The application*

$$(\cdot)^\wedge : \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n) \to \mathcal{L}(\mathcal{S}_n, \mathcal{S}_m)$$

is a module isomorphism.

Proof. It follows easily from the above theorem. ■

5 $\mathcal{O}_M$ Functions and Schwartz basis

In this section we study some important relations among a Schwartz family w and its multiples fw.

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a Schwartz family and let $f \in \mathcal{O}_M^{(m)}$. Then, the hull ${}^{\mathcal{S}}\mathrm{span}(w)$ of the family w contains the hull ${}^{\mathcal{S}}\mathrm{span}(fw)$ of the multiple family fw. Moreover, if a distribution a represents the distribution u with respect to the family fw (that is, if $u = a.(fw)$) then the distribution fa represents the distribution u with respect to the family w (that is, if $u = (fa).w$).*

Proof. 1) Let u be a vector of the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(fw)$. Then, there exists a coefficient distribution $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} a(fw),$$

and this is equivalent (as we already have seen) to the equality

$$u = \int_{\mathbb{R}^m} (fa)w;$$

hence the vector u belongs also to the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$. Hence the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(fw)$ is contained in the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$. ■

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a Schwartz family and let $f \in \mathcal{O}_M^{(m)}$ be a function different from 0 at every point of its domain. Then, the following assertions hold true:*

1) *if the family w is ${}^{\mathcal{S}}$linearly independent, the family fw is ${}^{\mathcal{S}}$linearly independent too;*

2) *the Schwartz linear hull ${}^{\mathcal{S}}\mathrm{span}(w)$ contains the hull ${}^{\mathcal{S}}\mathrm{span}(fw)$;*

3) *if the family w is ${}^{\mathcal{S}}$linearly independent, for each vector u in the hull ${}^{\mathcal{S}}\mathrm{span}(fw)$, we have*
$$[u \mid w] = f[u \mid fw],$$
where, as usual, by $[u|v]$ we denote the Schwartz coordinate system of a distribution u (in the Schwartz linear hull of v) with respect to a Schwartz linear independent family v;

4) *if the family w is an ${}^{\mathcal{S}}$basis of a subspace V, then fw is an ${}^{\mathcal{S}}$basis of its ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(fw)$ (that in general is a proper subspace of the hull ${}^{\mathcal{S}}\mathrm{span}(w)$).*

Proof. 1) Let $a \in \mathcal{S}'_m$ be such that
$$\int_{\mathbb{R}^m} a(fw) = 0_{\mathcal{S}'_n},$$
we have
$$\begin{aligned} 0_{\mathcal{S}'_n} &= \int_{\mathbb{R}^m} a(fw) = \\ &= \int_{\mathbb{R}^m} (fa)w, \end{aligned}$$
thus, because the family w is ${}^{\mathcal{S}}$linearly independent we have $fa = 0_{\mathcal{S}'_n}$. Since f is different from 0 at every point, we can conclude $a = 0_{\mathcal{S}'_n}$.

2) Let u be a vector of the Schwartz linear hull ${}^{\mathcal{S}}\mathrm{span}(fw)$. Then, there exists a coefficient distribution $a \in \mathcal{S}'_m$ such that
$$u = \int_{\mathbb{R}^m} a(fw),$$
or equivalently such that
$$u = \int_{\mathbb{R}^m} (fa)w,$$
and hence the vector u belongs also to the hull ${}^{\mathcal{S}}\mathrm{span}(w)$. Hence the Schwartz linear hull ${}^{\mathcal{S}}\mathrm{span}(fw)$ is contained in the hull ${}^{\mathcal{S}}\mathrm{span}(w)$.

3) If the family w is ${}^{\mathcal{S}}$linearly independent, from the above two equalities, we deduce $(u)_{fw} = a$ and $(u)_w = fa$, from which

$$\begin{aligned} (u)_w &= fa = \\ &= f(u)_{fw}, \end{aligned}$$

as we claimed.

4) is an obvious consequence of the preceding properties. ■

6 ${}^{\mathcal{O}_M}$Invertible functions and ${}^{\mathcal{S}}$basis

We recall that an invertible element of $\mathcal{O}_M^{(m)}$ is any function f everywhere different from 0 and such that its multiplicative inverse f^{-1} lives in $\mathcal{O}_M^{(m)}$ too. The set of the invertible elements of the space $\mathcal{O}_M^{(m)}$ is a group with respect to the pointwise multiplication, and we will denote it by $\mathcal{G}_M^{(m)}$.

Theorem. *Let $w \in \mathcal{S}(\mathbb{R}^m, \mathcal{S}'_n)$ be a Schwartz family and let $f \in \mathcal{G}_M^{(m)}$ be an invertible element of the ring $\mathcal{O}_M^{(m)}$ (in particular, it must be a function different form 0 at every point). Then, the following assertions hold true:*

1) *the family w is ${}^{\mathcal{S}}$linearly independent if and only if the multiple family fw is ${}^{\mathcal{S}}$linearly independent;*

2) *the hull ${}^{\mathcal{S}}\mathrm{span}(w)$ coincides with the hull ${}^{\mathcal{S}}\mathrm{span}(fw)$;*

3) *if the family w is ${}^{\mathcal{S}}$linearly independent, then, for each vector u in the hull ${}^{\mathcal{S}}\mathrm{span}(w)$, we have*

$$[u \mid fw] = (1/f)\,[u \mid w],$$

where, as usual, by $[u|v]$ we denote the Schwartz coordinate system of a distribution u (in the Schwartz linear hull of v) with respect to a Schwartz linear independent family v;

4) *the family w is an ${}^{\mathcal{S}}$basis of a subspace V if and only if its multiple fw is an ${}^{\mathcal{S}}$basis of the ${}^{\mathcal{S}}$linear hull ${}^{\mathcal{S}}\mathrm{span}(fw)$ (that in this case coincides with ${}^{\mathcal{S}}\mathrm{span}(w)$).*

Proof. 1) Let $a \in \mathcal{S}'_m$ be a distribution such that

$$\int_{\mathbb{R}^m} aw = 0_{\mathcal{S}'_n},$$

we have

$$\begin{aligned} 0_{\mathcal{S}'_n} &= \int_{\mathbb{R}^m} aw = \\ &= \int_{\mathbb{R}^m} (f^{-1}a)(fw), \end{aligned}$$

thus, because fw is ${}^{\mathcal{S}}$linearly independent we have $f^{-1}a = 0_{\mathcal{S}'_n}$. Since f^{-1} is different form 0 at every point we can conclude $a = 0_{\mathcal{S}'_n}$.

2) Let u be in ${}^{\mathcal{S}}\mathrm{span}(w)$. Then, there exists a distribution $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} aw.$$

Now, we have

$$u = \int_{\mathbb{R}^m} (f^{-1}a)\,(fw)\,,$$

so the distribution u lies in ${}^{\mathcal{S}}\mathrm{span}(fw)$, and hence ${}^{\mathcal{S}}\mathrm{span}(w)$ is contained in ${}^{\mathcal{S}}\mathrm{span}(fw)$. Vice versa, let u be in ${}^{\mathcal{S}}\mathrm{span}(fw)$. Then, there exists a distribution $a \in \mathcal{S}'_m$ such that

$$u = \int_{\mathbb{R}^m} a(fw).$$

Now, we have (equivalently)

$$u = \int_{\mathbb{R}^m} (fa)w,$$

and hence u lies also in ${}^{\mathcal{S}}\mathrm{span}(w)$, hence ${}^{\mathcal{S}}\mathrm{span}(fw)$ is contained in ${}^{\mathcal{S}}\mathrm{span}(w)$ (as we already have seen in the general case). Concluding

$${}^{\mathcal{S}}\mathrm{span}(w) = {}^{\mathcal{S}}\mathrm{span}(fw).$$

3) For any distribution u in the Schwartz linear hull of the family w, we have

$$u = \int_{\mathbb{R}^m} [u \mid w]\, w,$$

hence

$$u = \int_{\mathbb{R}^m} (f^{-1}[u|w])\,(fw)\,,$$

as we desired.

4) It follows immediately from the above properties. ■

Theorem. *Let $e \in \mathcal{B}(\mathbb{R}^m, \mathcal{S}'_n)$ be an ${}^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$ and let $f \in \mathcal{O}_M^{(m)}$. Then the multiple fe is an ${}^{\mathcal{S}}$basis of the space $\mathcal{S}'_n$ if and only if the factor f is an invertible element of the ring $\mathcal{O}_M^{(m)}$.*

Proof. We must prove that, if fe is an ${}^{\mathcal{S}}$basis of $\mathcal{S}'_n$, then f is an invertible element of the ring $\mathcal{O}_M^{(m)}$. First of all observe that, since fe is a basis, then fe is ${}^{\mathcal{S}}$linearly independent and consequently linearly independent in the ordinary algebraic sense; consequently every distribution $f(p)e_p$ must be a non zero distribution and this implies that any value $f(p)$ must be different from 0, so we can consider the multiplicative inverse f^{-1}. We now have to prove that the multiplicative inverse f^{-1} lives in $\mathcal{O}_M^{(m)}$, or equivalently that, for every test function g in $\mathcal{S}_m$, the product $f^{-1}g$ lives in $\mathcal{S}_m$. For, let g be in $\mathcal{S}_m$, since fe is a basis, its associated operator from $\mathcal{S}_n$ into $\mathcal{S}_m$ is surjective, then there is a function h in $\mathcal{S}_n$ such that $(fe)^\wedge(h) = g$, the last equality is equivalent to

$$fe(h) = g,$$

that is

$$f^{-1}g = e(h),$$

so that $f^{-1}g$ actually lives in the space $\mathcal{S}_m$. ■

We can generalize the above result as it follows.

Theorem. *Let $e \in \mathcal{B}(\mathbb{R}^m, V)$ be an ${}^{\mathcal{S}}$basis of a (weakly*) closed subspace V of the space $\mathcal{S}'_n$ and let $f \in \mathcal{O}_M^{(m)}$. Then the multiple family fe is an ${}^{\mathcal{S}}$basis of the subspace V if and only if the factor f is an invertible element of the ring $\mathcal{O}_M^{(m)}$.*

Proof. We must prove that, if fe is an ${}^{\mathcal{S}}$basis of the subspace V, then f is an invertible element of the ring $\mathcal{O}_M^{(m)}$. First of all observe that, since fe is a basis, then fe is ${}^{\mathcal{S}}$linearly independent and consequently linearly independent in the ordinary algebraic sense; consequently every distribution $f(p)e_p$ must be a non zero distribution and this implies that any value $f(p)$ must be different from 0. So we can consider its multiplicative inverse f^{-1}. We now have to prove that the multiplicative inverse f^{-1} lives in the space $\mathcal{O}_M^{(m)}$, or equivalently that, for every test function g in $\mathcal{S}_m$ the product $f^{-1}g$ lives in $\mathcal{S}_m$. For, let g be in $\mathcal{S}_m$, since fe is an ${}^{\mathcal{S}}$basis of the topologically closed subspace V, its associated operator $(fe)^\wedge$ from $\mathcal{S}_n$ into $\mathcal{S}_m$ is surjective (this follows, by the closedness of V, from the Dieudonné-Schwartz theorem, since the transpose of the operator $(fe)^\wedge$ is the superposition operator of fe, which is injective since the family fe is Schwartz linearly independent). Hence, by surjectivity, there is a function h in $\mathcal{S}_n$ such that $(fe)^\wedge(h) = g$, the last equality is equivalent to the following one

$$fe(h) = g,$$

that is

$$f^{-1}g = e(h),$$

so that the function $f^{-1}g$ actually lives in the space $\mathcal{S}_m$. ■

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